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Band invariants for perturbations of the harmonic oscillator

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Abstract

We study the direct and inverse spectral problems for semiclassical operators of the form $S = S_0 + \hbar^2 V$, where $S_0 = \frac{1}{2}(-\hbar^2 \Delta_{\mathbb{R}^n} + |x|^2)$ is the harmonic oscillator and $V : \mathbb{R}^n \to \mathbb{R}$ is a tempered smooth function. We show that the spectrum of S forms eigenvalue clusters as $\hbar$ tends to zero, and compute the first two associated "band invariants". We derive several inverse spectral results for V, under various assumptions. In particular we prove that, in two dimensions, generic analytic potentials that are even with respect to each variable are spectrally determined (up to a rotation).
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Contents

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1. Introduction

Consider the semiclassical harmonic oscillator

$$S_0 = \frac{1}{2}\left(-\hbar^2\Delta_{\mathbb{R}^n} + |x|^2 - \hbar n I\right)$$

acting on $L^2(\mathbb{R}^n)$, where, to simplify the notation throughout the paper, we have subtracted the ground state energy. The spectrum of S_0 consists of the eigenvalues

$$E_j(\hbar) = \hbar j, \quad j = 0, 1, 2, \ldots, \tag{1.1}$$

with multiplicity

$$m_j = \binom{n+j-1}{n-1}.$$

We will study perturbations of S_0 of the form

$$S = S_0 + \hbar^2 A, \tag{1.2}$$

where A is a self-adjoint semiclassical pseudodifferential operator of order zero. Since the distance between consecutive eigenvalues of S_0 is $O(\hbar)$, if A is L^2 bounded for $\hbar$ sufficiently small the spectrum of S forms non-overlapping clusters of eigenvalues

$$\{E_{j,k};\ k = 1, \ldots, m_j\}, \quad j = 0, 1, \ldots,$$

defined by the condition

$$\left|E_{j,k}(\hbar) - E_j(\hbar)\right| = O\left(\hbar^2\right).$$

As we will see the L^2 boundedness assumption of A can be dropped, provided one looks at the spectrum of S locally: For all A in standard symbol classes the spectrum of S restricted to compact intervals forms spectral clusters for small enough $\hbar$.

By analyzing these spectral clusters using wave-trace techniques, one generates a collection of spectral invariants which are known as "band invariants" and have been studied in detail in various contexts in [1,15,3,4,2,13,14,16], and elsewhere. The main topic of this paper will be computations of these invariants for the harmonic oscillator. These computations will be valid in arbitrary dimension, but in dimension two we will be able to extract from them a number of new inverse results. In particular, for the Schrödinger-like operator $S_0+\hbar^2 V$, we will show that if V is real analytic and symmetric in its x_1 and x_2 coordinates, and if, in addition, the two eigenvalues of the Hessian of V at the origin are distinct, then V is spectrally determined. In fact we will show more generally that a result of this nature is true for real analytic semiclassical potentials, $V = V(x,\hbar)$, and for C^∞ semiclassical potentials provided that $V(x,0)$ is a quadratic form with distinct eigenvalues. (See Section 6 for details.)

Perturbations of the form $S_0 + \hbar^{1+\delta} A$, $\delta > 0$ and A a bounded zeroth order pseudodifferential operator, have been studied recently by D. Ojeda-Valencia and C. Villegas Blas [12]. Working in Bargmann space they derive the first band invariant for such perturbations.

2. The clustering phenomenon

We will be working with semiclassical pseudodifferential operators, with amplitudes in the symbol classes $\mathcal{O}(\langle x,\xi\rangle^m)$, and using the Weyl calculus (see [11] for definitions).

2.1. Existence of spectral clusters

Let S be as in (1.2). The following theorem is a special case of a result by B. Helffer and D. Robert in [9] (Theorem 3.9). It states that, if we only look at eigenvalues contained in a fixed "window", then as $\hbar \to 0$, the eigenvalues of S form clusters around the unperturbed eigenvalues. We include a proof here for completeness, and to set up the computation of the band invariants.

Theorem 2.1. *For any compact interval $I \subset \mathbb{R}$ there exists $\hbar_0 > 0$ small enough such that for $0 < \hbar < \hbar_0$, the spectrum of S in I consists of eigenvalues that cluster near the eigenvalues of S_0 within a distance $O(\hbar^2)$. More precisely, given I*

$$\begin{aligned} &\exists C > 0,\ \hbar_0 > 0 \quad \textit{such that} \quad \forall h \in (0,\hbar_0],\\ &\operatorname{Spec}(S) \cap I \subset \bigcup_{j\in\mathbb{Z}_+} \left[\hbar j - C\hbar^2, \hbar j + C\hbar^2\right]. \end{aligned} \tag{2.1}$$

Proof. Define the time-dependent operator $R(t)$ by the identity

$$e^{i\hbar^{-1}tS_0} e^{-i\hbar^{-1}tS} = I + \hbar R(t). \tag{2.2}$$

The left-hand side of this identity is a semiclassical pseudodifferential operator (a composition of semiclassical FIOs with inverse canonical relations), so $R(t)$ is a semiclassical pseudodifferential operator. In what follows we let

$$R = R(2\pi) = \frac{1}{\hbar}\left(e^{-2\pi i \hbar^{-1} S} - I\right).$$

Now let $\chi \in C_0^\infty(\mathbb{R})$ be a cut-off function which is identically equal to one in an open neighborhood of I, and define

$$R_\chi = \chi(S) R.$$

This is a semiclassical pseudodifferential operator, and in the expansion of its symbol

$$r^\chi(x, p, \hbar) \sim \sum \hbar^j r_j^\chi(x, p),$$

one can see that each r_j^χ has compact support contained in $H_0^{-1}(\mathrm{supp}(\chi))$, where

$$H_0(x, p) = \frac{1}{2}\left(|x|^2 + |p|^2\right) \tag{2.3}$$

is the principal symbol of S_0. It follows from the Calderón–Vaillancourt theorem that R_χ is a bounded operator in L^2, and the bound is *uniform* in $\hbar$. Thus, for $\hbar$ sufficiently small the spectrum of $I + \hbar R_\chi$ is contained in a neighborhood of 1 away from the origin in $\mathbb{C}$, and we can define

$$W = \frac{i}{2\pi\hbar} \log(I + \hbar R_\chi), \tag{2.4}$$

which is a bounded pseudodifferential operator of order zero. Define now

$$S_\chi := S - \hbar^2 W.$$

Note that all the operators in this paragraph are functions of S, and therefore commute with each other.

It follows from the definitions that

$$e^{-2\pi i \hbar^{-1} S_\chi} = (I + \hbar R)(I + \hbar R_\chi)^{-1}.$$

Now let $\Lambda \in I$ be an eigenvalue of S with normalized eigenvector ψ. Since

$$\chi(S)(\psi) = \psi,$$

it is clear that

$$e^{-2\pi i \hbar^{-1} S_\chi}(\psi) = \psi,$$

which is to say that $S_\chi \psi = \hbar j \psi$ for some integer j. From this and the fact that $S = S_\chi + \hbar^2 W$ it follows that

$$\Lambda = \hbar j + \hbar^2 \mu, \quad \text{where } W\psi = \mu\psi. \tag{2.5}$$

Therefore

$$|\Lambda - \hbar j| \leqslant C\hbar^2,$$

where C is a uniform L^2 bound on the norm of W for $\hbar$ small enough. This completes the proof. □

Note that the intervals on the right-hand side of (2.1) are pairwise disjoint for $\hbar$ small enough, since their centers are a distance $\hbar$ apart and their lengths are $O(\hbar^2)$. The eigenvalues of S in any one of those disjoint intervals must be of the form

$$E_{j,k} = E_j + \hbar^2 \mu_{j,k}, \tag{2.6}$$

where the $\mu_{j,k}$ are the eigenvalues of W restricted to the eigenspace of S_χ corresponding to $E_j = \hbar j$. Note that the $\mu_{j,k}$ are uniformly bounded.

Remark 2.2. The previous theorem holds for more general *Zoll operators*. Namely, let Q be a semiclassical pseudodifferential operator such that $\operatorname{Spec}(Q) \subset \hbar\mathbb{Z}$. (This implies that the Hamilton flow of its principal symbol is 2π periodic.) Let A be a zeroth order semiclassical pseudodifferential operator whose symbol lies in $\mathcal{O}(\langle x, \xi\rangle^m)$, and consider the perturbation of Q

$$P = Q + \hbar^2 A.$$

Then for any compact interval $I \subset \mathbb{R}$ the eigenvalues of P in I cluster near $\hbar\mathbb{Z}$, for all $\hbar$ small enough.

2.1.1. Clusters for potentials of quadratic growth

In the case that the perturbation A is a "multiplication by a potential function $V(x)$" operator, there is a much simpler and direct proof of the previous result provided V is of no more than quadratic growth at infinity. Moreover, in this case one can see the spectral clustering appearing uniformly in all of $\mathbb{R}$, instead only locally uniformly in compact intervals. In other words, there exists $C > 0$ and $\hbar_0 > 0$ such that for all $\hbar < \hbar_0$,

$$\operatorname{Spec}\big(S_0 + \hbar^2 V\big) \subset \bigcup_{k \in \mathbb{Z}_+} \big[\hbar k - C\hbar^2, \hbar k + C\hbar^2\big]. \tag{2.7}$$

Here is a sketch of the proof: If V is of no more than quadratic growth at infinity, then one can find constants $C_1 > 0$, $C_2 > 0$ such that

$$-C_1|x|^2 - C_2 < V(x) < C_1|x|^2 + C_2.$$

On the other hand, by directly computations, one can show that the eigenvalues of

$$S_0 + \hbar^2\big(\pm C_1|x|^2 \pm C_2\big) = -\frac{1}{2}\hbar^2\Delta + \frac{1}{2}|x|^2\big(1 \pm 2\hbar^2 C_1\big) \pm \hbar^2 C_2$$

are precisely $\hbar\sqrt{1 \pm 2\hbar^2 C_1}(j + \frac{n}{2}) \pm \hbar^2 C_2$, which are within $O(\hbar^2)$ distance of the unperturbed eigenvalues (1.1). Now (2.7) follows from a min–max argument.

Remark 2.3. Analogous arguments show that the same result holds for a "semiclassical potential",

$$A = V_0(x) + \hbar V_1(x) + \hbar^2 V_2(x) + \cdots,$$

where all the V_j are of functions with uniform quadratic growth: $|V_j(x)| < C_1 + C_2|x|^2$ (for some constants C_1, C_2 independent of j).

2.2. Cluster projectors

Let $I \subset \mathbb{R}$ be a compact interval. Keeping the notations of Theorem 2.1, let $E \in \mathrm{Int}(I)$, $E > 0$, and restrict the values of $\hbar$ to the sequence

$$\hbar = \frac{E}{N}, \quad N = 1,\ 2, \ldots. \tag{2.8}$$

As we have seen in Theorem 2.1, there exists $C > 0$ so that for each such $\hbar$ small enough,

$$\mathrm{Spec}(S) \cap \big[E - C\hbar^2, E + C\hbar^2\big] = \big\{E + h^2\mu_{N,k};\ k = 1, \ldots, m_N\big\} \tag{2.9}$$

is a single cluster (see (2.6)).

Theorem 2.4. *For each $\hbar$ as in (2.8), let Π_E^N be the orthogonal projection from $L^2(\mathbb{R}^n)$ onto the span of the eigenfunctions of S corresponding to eigenvalues in the individual cluster (2.9). Then the family $\{\Pi_E^N\}_N$ is a semiclassical Fourier integral operator associated with the canonical relation*

$$\big\{(\bar{x}, \bar{y}) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}\ \big|\ H_0(\bar{x}) = E = H_0(\bar{y}),\ \bar{x} \text{ lies on the same } H_0 \text{ orbit as } \bar{y}\big\}.$$

Proof. Take a closed interval I_1 containing E in its interior and included in $\mathrm{Int}(I)$, and a cutoff function χ_1 such that $\chi_1 = 1$ on I_1 and $\chi_1 = 0$ outside I. Define

$$U(t) = \chi_1(S_\chi)e^{-2\pi i\hbar^{-1}tS_\chi}.$$

Then one can check that (recalling that $\hbar = E/N$),

$$\frac{1}{2\pi}\int_0^{2\pi} U(t)e^{it\hbar^{-1}E}\,dt = \chi_1(S_\chi)\Pi_E^N. \tag{2.10}$$

Now $U(t)$ is a Fourier integral operator associated with the graph of the principal symbol of S_χ. The conclusion of the theorem follows from this and the calculus of FIOs applied to the left-hand side of (2.10). $\square$

3. Band invariants

3.1. The first band invariant

According to Eq. (2.2), the time derivative of $R(t)$ is given by

$$\dot{R}(t) = \frac{i}{\hbar}\big[S_0, R(t)\big] - iA - i\hbar A R(t). \tag{3.1}$$

It follows that its principal symbol, $r_0(t)$, satisfies

$$\dot{r}_0(t) = \{H_0, r_0\} - ia \quad \text{and} \quad r_0|_{t=0} = 0,$$

where a is the principal symbol of A. The solution to this problem is

$$r_0(t, x, p) = -i\int_0^t a\big(\phi_s(x, p)\big)\, ds,$$

where ϕ_s is the Hamilton flow of H_0, i.e. $\phi_s(x, p)$ is the time-s solution $(x(s), p(s))$ to the Hamilton system

$$\dot{x}_j(t) = p_j(t), \qquad \dot{p}_j(t) = -x_j(t)$$

with initial conditions $x(0) = x$, $p(0) = p$.

In what follows we will denote, for any function b on $\mathbb{R}^{2n}$,

$$b^{\mathrm{ave}}(x, p) := \frac{1}{2\pi}\int_0^{2\pi} b\big(\phi_s(x, p)\big)\, ds. \tag{3.2}$$

With this notation, the principal symbol of

$$W = \frac{i}{2\pi\hbar}\log(I + \hbar R_\chi)$$

is a^{ave}.

The first band invariant arises as follows:

Theorem 3.1. *For any $f \in C_0^\infty(\mathbb{R})$ and $\varphi \in C^\infty(\mathbb{R})$, the integral*

$$\int f\left(\frac{|x|^2 + |p|^2}{2}\right)\varphi\big(a^{\mathrm{ave}}\big)\, dx\, dp \tag{3.3}$$

is a spectral invariant for the family of operators (1.2).

Proof. Choose the interval I in Theorem 2.1 such that supp $f \Subset I$. Then

$$\sum_j f(E_j) \sum_{k=1}^{m_j} \varphi(\mu_{k,j}) = \mathrm{Tr}\big[f(S_\chi)\varphi(W)\big].$$

As $\hbar \to 0$, the trace on the right-hand side has an asymptotic expansion with leading coefficient (3.3). □

As a consequence, one can see that for any $E > 0$, the numbers

$$\int\limits_{|x|^2+|p|^2=E} \varphi\big(a^{\mathrm{ave}}\big)\, d\lambda \tag{3.4}$$

are spectral invariants of the semiclassical family of operators $S = S_0 + \hbar^2 A$, where $d\lambda$ is the (normalized) standard Lebesgue measure on the sphere $|x|^2 + |p|^2 = E$. In fact, these numbers arise as Szegö limits:

Theorem 3.2. *Fix $E > 0$ and $\varphi \in C^\infty(\mathbb{R})$. Then, as $N \to \infty$*

$$\frac{1}{m_N} \sum_{k=1}^{m_N} \varphi(\mu_{N,k}) = \int\limits_{|x|^2+|p|^2=E} \varphi\big(a^{\mathrm{ave}}\big)\, d\lambda + O(1/N). \tag{3.5}$$

The proof is standard: The left-hand side of (3.5) is the normalized trace of $\varphi(W)\Pi_E^N$. The asymptotic behavior of this trace can be computed symbolically given that the cluster projectors are FIOs (Theorem 2.4).

3.2. Some properties of the averaging procedure

For future reference we gather here some properties of the averaging procedure, restricted to functions V of the position variable x alone. For such functions, letting $z = x + ip$, one can rewrite the integral (3.2) as

$$V^{\mathrm{ave}}(z) = \frac{1}{2\pi} \int_0^{2\pi} V\left(\frac{e^{it}z + e^{-it}\bar{z}}{2}\right) dt.$$

From this integral formula one can derive many properties of V^{ave} as a function on $\mathbb{C}^n$. Here is a very incomplete list:

(1) V^{ave} is always an even function. Moreover, one can regard V^{ave} as a function defined on $S^{2n-1}/S^1 = \mathbb{CP}^{n-1}$.
(2) If $V_1(x) = V_2(x)$ on a ball $|x|^2 \leqslant r^2$, then $V_1^{\mathrm{ave}} = V_2^{\mathrm{ave}}$ on the complex ball $|z|^2 \leqslant r^2$.

(3) If $V(x)$ is an odd function, then $V^{\mathrm{ave}} \equiv 0$. (In later applications, we will mainly focus on the V's that are even in *each* variable.)

(4) If $V(x)$ is of polynomial growth,

$$\left|V(x)\right| \leqslant C_1 + C_2|x|^m,$$

so is $V^{\mathrm{ave}}(z)$, i.e.

$$\left|V^{\mathrm{ave}}(z)\right| \leqslant C_1 + C_2|z|^m.$$

More generally, if $|V(x)| \leqslant h(|x|)$ for some *increasing* function h, then $|V^{\mathrm{ave}}(z)| \leqslant h(|z|)$.

(4a) In general, one cannot drop the "increasing" assumption on h above. For example, consider $n = 1$ and $V(x) = \frac{1}{1+|x|^2}$, then $V(x) = O(\frac{1}{|x|^2})$ as $x \to \infty$, and

$$V^{\mathrm{ave}}(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1+\left(\frac{e^{it}z+e^{-it}\bar{z}}{2}\right)^2}\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1+\frac{|z|^2}{2}\big(1+\cos(2t)\big)}\,dt.$$

One can check $V^{\mathrm{ave}}(z) \neq O(\frac{1}{|z|^2})$.

(4b) However, if $V(x) = o(1)$, then $V(z) = o(1)$. We will leave the proof as an exercise.

(4c) Similarly, if $V(x) \geqslant C|x|^N$ for $|x|$ large enough, then $V^{\mathrm{ave}}(z) \geqslant C'|z|^N$ for $|z|$ large enough.

(5) If $V(x)$ is homogeneous of degree m in x, then $V^{\mathrm{ave}}(z)$ is also homogeneous of degree m in z.

(6) If $V(x) = x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$ is a monomial, then V^{ave} is zero unless $|\alpha|$ is even, in which case

$$V^{\mathrm{ave}}(z) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{e^{it}z_1+e^{-it}\bar{z}_1}{2}\right)^{\alpha_1}\cdots\left(\frac{e^{it}z_n+e^{-it}\bar{z}_n}{2}\right)^{\alpha_n}dt$$

$$= \frac{1}{2^{|\alpha|}}\sum_{\substack{j_1+\cdots+j_n=\frac{|\alpha|}{2}\\ \alpha_r\geqslant j_r\geqslant 0}}\binom{\alpha_1}{j_1}\cdots\binom{\alpha_n}{j_n} z_1^{j_1}\bar{z}_1^{\alpha_1-j_1}\cdots z_n^{j_n}\bar{z}_n^{\alpha_n-j_n}. \tag{3.6}$$

(7) V^{ave} is identically zero if and only if V is an odd function. We will prove this in the $n = 2$ case in Section 5. The case $n > 2$ reduces to the two dimensional case, as the trajectories of the harmonic oscillator in any dimension lie on two dimensional planes.

3.3. Higher band invariants for perturbations by potentials

To derive higher order band invariants, one must look at higher order terms in the symbol of W. Recall that by the semiclassical Weyl calculus, the Weyl symbol of a composition PQ is

$$\begin{aligned} a\#_\hbar b &\sim e^{\frac{i\hbar}{2}(D_x D_q - D_y D_p)}\big(a(x,p)b(y,q)\big)\big|_{y=x,q=p} \\ &= \sum \hbar^j B_j(a,b) \end{aligned}$$

for some bi-differential operators B_j, where

$$B_0(a,b) = ab \quad \text{and} \quad B_1(a,b) = \frac{1}{2i}\{a,b\}.$$

In particular, the Weyl symbol of the commutator $\frac{i}{\hbar}[S_0, R]$ is given by

$$\{H_0, r\} + \hbar^2\{H_0, r\}_3 + \hbar^4\{H_0, r\}_5 + \cdots,$$

where $\{\ ,\ \}$ is the Poisson bracket, and $\{\ ,\ \}_k$ is the "higher Poisson bracket"

$$\{a,b\}_k = \sum_{|\alpha|+|\beta|=k} \frac{1}{\alpha!\beta!}(-1)^{|\alpha|}\partial_p^\beta \partial_x^\alpha a \partial_p^\alpha \partial_x^\beta b.$$

Let us now assume that the operator A is the operator "multiplication by a potential function $V(x)$", i.e.

$$S = S_0 + \hbar^2 V. \tag{3.7}$$

Since $H_0 = \frac{1}{2}(|x|^2 + |p|^2)$ is quadratic in both x and p, all higher order Poisson brackets of H_0 with $r(t)$ vanish. If we write

$$r(t,\hbar,x,p) \sim r_0(t,x,p) + \hbar r_1(t,x,p) + \hbar^2 r_2(t,x,p) + \cdots,$$

and if abbreviate $r_k(t,x,p)$ to $r_k(t)$, then from (3.1) we get

$$\sum_k \hbar^k \dot r_k(t) = \sum_k \hbar^k \big\{H_0, r_k(t)\big\} - iV - i\sum_{j,l} \hbar^{j+l+1} B_l(V, r_j)$$

with initial conditions $r_k(0) = 0$. In particular,

$$\dot r_1(t) = \{H_0, r_1\} - iVr_0 \quad \text{and} \quad r_1|_{t=0} = 0.$$

It is easy to check that the solution to this equation is simply

$$r_1(t,x,p) = \frac{1}{2} r_0^2(t,x,p).$$

Similarly the equation for r_2 is

$$\dot{r}_2(t) = \{H_0, r_2\} - i\big(Vr_1 + B_1(V, r_0)\big) \quad \text{and} \quad r_1|_{t=0} = 0,$$

and its solution is

$$r_2(t, x, p) = \frac{1}{6}r_0^3 + \frac{i}{2}\int_0^t \int_0^{t-s} \big\{V\big(\phi_s(x, p)\big), V\big(\phi_{s+u}(x, p)\big)\big\}\, du\, ds.$$

In general, for any $k \geqslant 1$, the $r_k(t, x, p)$ is the solution to the problem

$$\dot{r}_k(t) = \{H_0, r_k\} - i\sum_{l=0}^{k-1} B_l(V, r_{k-1-l}) \quad \text{and} \quad r_k|_{t=0} = 0. \tag{3.8}$$

Note that if we let

$$g_k(t) = \sum_{l=0}^{k-1} B_l(V, r_{k-1-l}),$$

then g_k depends only on $r_0, \dots, r_{k-1}$, in particular it is independent of r_k. So the problem (3.8) can be solved via Duhamel's principle iteratively:

$$r_k(t, x, p) = -i\int_0^t g_k\big(t - s, \phi_s(x, p)\big)\, ds.$$

Replacing t by 2π, we get the higher-order symbols of R, and thus the symbol of R_χ in the region $H_0^{-1}(I)$.

To calculate the higher-order symbols of W, note that

$$\begin{aligned} W &= \frac{i}{2\pi\hbar}\log(I + \hbar R_\chi) \\ &= \frac{i}{2\pi}\left(R_\chi - \frac{\hbar}{2}R_\chi^2 + \frac{\hbar^2}{3}R_\chi^3 + \cdots\right). \end{aligned}$$

So if we let

$$w \sim w_0 + \hbar w_1 + \cdots$$

be the full symbol of W, then $w_0 = V^{\mathrm{ave}}$, and

$$w_k = \frac{1}{2\pi i}\sum_{j=1}^{k+1} \frac{(-1)^{j+1}}{j} \sum_{l_1+\cdots+l_{j-1}+m_1+\cdots+m_j = k-j+1} B^{l_1,\dots,l_{j-1}}(r_{m_1}, \dots, r_{m_j}),$$

where r_j's are symbols of R_χ calculated above, all indices l_j, m_j's are non-negative, and

$$B^{l_1,\ldots,l_{j-1}}(r_{m_1},\ldots,r_{m_j}) = B_{l_{j-1}}\big(B_{l_{j-2}}\big(\cdots\big(B_{l_1}(r_{m_1},r_{m_2}),r_{m_3}\big)\cdots\big),r_{m_{j-1}},r_{m_j}\big).$$

In particular,

$$w_1 = r_1 - \frac{1}{2}r_0^2 = 0 \tag{3.9}$$

and

$$\begin{aligned} w_2 &= \frac{i}{2\pi}\left(r_2 - r_0 r_1 + \frac{1}{3}r_0^3\right) \\ &= -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi-s}\big\{V\big(\phi_s(x,p)\big),V\big(\phi_{s+u}(x,p)\big)\big\}\,du\,ds \\ &= -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{u}\big\{V\big(\phi_s(x,p)\big),V\big(\phi_u(x,p)\big)\big\}\,ds\,du. \end{aligned}$$

In what follows we will let, for any function F on phase space,

$$F^\Delta(x,p) := -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{u}\big\{F\big(\phi_s(x,p)\big),F\big(\phi_u(x,p)\big)\big\}\,ds\,du, \tag{3.10}$$

so that

$$w_2 = V^\Delta. \tag{3.11}$$

We are now in a position to extend Theorem 3.1 and obtain higher order terms in the asymptotic expansion of

$$\sum_j f(E_j)\sum_{k=1}^{m_j}\varphi(\mu_{k,j}) = \mathrm{Tr}\big[f(S_\chi)\varphi(W)\big]. \tag{3.12}$$

Theorem 3.3. *Let $f \in C_0^\infty(\mathbb{R})$ and $\varphi(s) = s^{l+1}$ in (3.12), where $l \geqslant 0$ is an integer. Then (3.12) has a semiclassical expansion in powers of $\hbar$, and the second coefficient is equal to*

$$(l+1)\int f\left(\frac{|x|^2+|p|^2}{2}\right)\big(V^{\mathrm{ave}}\big)^l V^\Delta\,dx\,dp + Q_l^f,$$

where Q_l^f is equal to the integral over $\mathbb{R}^{2n}$ of the following expression:

$$\begin{aligned} &B_2\big(f(H_0), \big(V^{\mathrm{ave}}\big)^{l+1}\big) + f(H_0)\big(V^{\mathrm{ave}}\big)^{[l]} \\ &\quad + \big(V^{\mathrm{ave}}\big)^{l+1}\big[f'(H_0)\big(V + V^{\mathrm{ave}}\big) - \mathcal{R}(f)(H_0)\big]. \end{aligned} \tag{3.13}$$

Here B_2 is the $\hbar^2$ operator in the Moyal product, we have let

$$w^{[l]} = \sum_{j=0}^{l-1} w^j B_2\big(w, w^{l-j}\big),$$

and $\mathcal{R}$ is the operator

$$\mathcal{R}(f)(s) = \frac{n}{8} f''(s) + \frac{s}{12} f'''(s).$$

The proof is a calculation that we have sketched in Appendix A.

3.4. The case of odd potentials

Recall that V is odd iff V^{ave} is identically zero. By Eqs. (3.9, 3.11), in this case the operator W is of order -2 and its principal symbol is V^{Δ}.

Theorem 3.4. *For odd potentials V, the integrals*

$$\int f\left(\frac{|x|^2 + |p|^2}{2}\right) \varphi\big(V^{\Delta}\big)\, dx\, dp$$

where $f \in C_0^\infty(\mathbb{R})$ and $\varphi \in C^\infty(\mathbb{R})$ are spectral invariants.

Proof. Analogously as in Theorem 3.1, the previous quantity is the coefficient of the leading order term in the asymptotic expansion of

$$\sum_j f(E_j) \sum_{k=1}^{m_j} \varphi\big(\hbar^{-2}\mu_{k,j}\big) = \mathrm{Tr}\big[f(S_\chi)\varphi\big(\hbar^{-2}W\big)\big].$$

In the present case $\hbar^{-2}W$ is a pseudodifferential operator of order zero and principal symbol V^{Δ}. $\square$

3.5. Perturbations by semiclassical potentials

More generally, we can consider the harmonic oscillator perturbed by a semiclassical potential, that is, consider

$$S = S_0 + \hbar^2 V(x, \hbar), \tag{3.14}$$

where

$$V(x,\hbar) \sim V_0(x) + \hbar V_1(x) + \hbar^2 V_2(x) + \cdots$$

as symbols in $\mathcal{O}(\langle x,\xi\rangle^m)$. The first band variant was calculated in Section 3.1, with $a(x,p) = V_0(x)$. For the higher invariants, a similar calculation as in Section 3.3 shows that for $k \geqslant 1$,

$$\dot{r}_k = -\{H_0, r_k\} - iV_k - i \sum_{m+n+l+1=k} B_l(V_m, r_n) \quad \text{and} \quad r_k|_{t=0} = 0.$$

Again these equations can be solved via Duhamel's principle. In particular, we get

$$r_1(t,x,p) = \frac{1}{2} r_0^2 + i \int_1^t V_1\big(\phi_s(x,p)\big)\, ds,$$

which leads to $w_1 = V_1^{\mathrm{ave}}$. More generally,

$$r_k(t,x,p) = i \int_0^t V_k\big(\phi_s(x,p)\big)\, ds + \text{terms depending only on } V_0, \dots, V_{k-1}$$

and

$$w_k = V_k^{\mathrm{ave}} + \text{terms depending only on } V_0, \dots, V_{k-1}.$$

So if we take $\varphi(x) = x^{l+1}$, then the $\hbar^{2l+k}$ term in

$$\sum_j f(E_j) \sum_{k=1}^{m_j} \varphi\big(\mu_{k,j}(\hbar)\big) = \mathrm{Tr}\big[f(S_\chi)\varphi(W)\big]$$

is

$$f(H_0)\big(V_0^{\mathrm{ave}}\big)^l V_k^{\mathrm{ave}} + \text{terms depending only on } V_0, \dots, V_{k-1}.$$

We conclude:

Theorem 3.5. *From the kth term of the expansion one can spectrally determine the quantities*

$$\int f\left(\frac{|x|^2 + |p|^2}{2}\right)\big(V_0^{\mathrm{ave}}\big)^l V_k^{\mathrm{ave}}\, dx\, dp + Q_l^f(V_0, \dots, V_{k-1}), \tag{3.15}$$

where $Q_l^f(V_0, \dots, V_{k-1})$ depends only on $V_0, \dots, V_{k-1}$ (and their derivatives) and on f, l.

4. First inverse spectral results

4.1. Spectral rigidity

Obviously for any $O \in SO(n)$, the rotated potential

$$V^O(x) := V(Ox)$$

is isospectral with the potential $V(x)$. From this observation one can easily construct trivial families of isospectral potentials.

Definition 4.1. We say a potential $V(x)$ is *spectrally rigid* if for any smooth family of isospectral potentials $V^t(x)$, with $V^0(x) = V(x)$, there is a smooth family of orthogonal matrices $O_t \in SO(n)$, such that $V^t(x) = V(O_t x)$.

Remark 4.2. Consider the Taylor expansion of V at the origin:

$$V(x) \sim V(0) + \sum \frac{\partial V}{\partial x_i}(0)x_i + \sum \frac{\partial^2 V}{\partial x_i \partial x_j}(0)x_i x_j + \text{higher order terms.}$$

It is obvious that if we rotate $V(x)$ to $V^O(x)$, the following remain unchanged:

- the constant term $V(0)$,
- the length $|\nabla V(0)|^2 = \sum(\frac{\partial V}{\partial x_i}(0))^2$,
- the eigenvalues of the quadratic form $\sum \frac{\partial^2 V}{\partial x_i \partial x_j}(0)x_i x_j$.

We will see that these data are spectrally determined in many cases. Moreover, it is easy to see that the effect that the constant term $V(0)$ and the linear term $\sum \frac{\partial V}{\partial x_i}(0)x_i$ have on the spectrum are simply translations, and therefore one may drop these terms and only consider potentials of the form

$$V(x) = \sum a_i x_i^2 + \text{higher order terms.}$$

(For a semiclassical potential, one may assume the leading term $V_0(x)$ if of this form.)

Remark 4.3. For a semiclassical potential one has another method to construct trivial isospectral families. Namely, suppose

$$V(x, \hbar) = V_0(x) + \hbar V_1(x) + \hbar^2 V_2(x) + \cdots$$

is a semiclassical potential. Then a change of variables

$$x_i \to x_i + b_i \hbar^2$$

converts the operator

$$\frac{1}{2}\left(-\hbar^2\Delta + |x|^2\right) + \hbar^2 V(x,\hbar)$$

into

$$\frac{1}{2}\left(-\hbar^2\Delta + |x|^2\right) + \hbar^2\left(V\left(x + b\hbar^2,\hbar\right) + \sum x_i b_i + \hbar^2\frac{b^2}{2}\right).$$

Therefore the semiclassical potential

$$\widetilde{V}(x,\hbar) = V\left(x + b\hbar^2,\hbar\right) + \sum x_i b_i + \hbar^2\frac{b^2}{2}$$

is isospectral with $V(x,\hbar)$. Note that if we take $b_i = -\frac{\partial V_0}{\partial x_i}(0)$, then the leading term of $\widetilde{V}(x,\hbar)$ is

$$\widetilde{V_0}(x) = V_0(x) - \sum \frac{\partial V_0}{\partial x_i}(0)x_i,$$

which has no linear term. Iteratively using this method, i.e. making changes of variables

$$x_i \to x_i + b_i\hbar^m,$$

$m \geqslant 2$, one can convert $V(x,\hbar)$ to an isospectral semiclassical potential

$$\widehat{V}(x,\hbar) = \widehat{V_0}(x) + \hbar\widehat{V_1}(x) + \hbar^2\widehat{V_2}(x) + \cdots$$

where each $\widehat{V_i}(x)$ has no linear term.

Now let $V(x)$ be an even potential function. We say that V is *formally spectrally rigid* if for any smooth family of isospectral even potentials, $V^t(x)$, with $V^0(x) = V(x)$, we have

$$\left.\frac{d^k}{dt^k}\right|_{t=0} V^t = 0 \tag{4.1}$$

for all $k \geqslant 1$.

Theorem 4.4. *Let $V(x)$ be a potential which is even in each variable. Then V is formally spectrally rigid if it satisfies*: *the only even function W such that*

$$\int f\left(|z|^2\right)\varphi\left(V^{\mathrm{ave}}\right)W^{\mathrm{ave}}\,dz\,d\bar{z} = 0 \tag{4.2}$$

holds for all $f \in C_0^\infty(\mathbb{R})$ and $\varphi \in C^\infty(\mathbb{R})$ is the zero function, $W = 0$.

Proof. Let $V^t(x)$ be a smooth family of isospectral even potentials, such that $V^0(x) = V(x)$. Then

$$0 = \frac{d}{dt}\bigg|_{t=0} \int f\big(|z|^2\big)\varphi\big(\big(V^t\big)^{\mathrm{ave}}\big)\, dz\, d\bar{z}$$
$$= \int f\big(|z|^2\big)\varphi'\big(V^{\mathrm{ave}}\big)\frac{d}{dt}\bigg|_{t=0}\big(V^t\big)^{\mathrm{ave}}\, dz\, d\bar{z},$$

which implies $\frac{d}{dt}|_{t=0}V^t = 0$. Similarly

$$0 = \frac{d^2}{dt^2}\bigg|_{t=0} \int f\big(|z|^2\big)\varphi\big(\big(V^t\big)^{\mathrm{ave}}\big)\, dz\, d\bar{z}$$
$$= \int f\big(|z|^2\big)\left[\varphi''\big(V^{\mathrm{ave}}\big)\left(\frac{d}{dt}\bigg|_{t=0}\big(V^t\big)^{\mathrm{ave}}\right)^2 + \varphi'\big(V^{\mathrm{ave}}\big)\frac{d^2}{dt^2}\bigg|_{t=0}\big(V^t\big)^{\mathrm{ave}}\right] dz\, d\bar{z},$$

which implies (4.1) for $k = 2$. Continuing in the same fashion, one can prove (4.1) for all k. □

Similarly one can define the *formally spectral rigidity* for semiclassical potentials

$$V(x, \hbar) = V_0(x) + \hbar V_1(x) + \hbar^2 V_2(x) + \cdots,$$

where each $V_k(x)$ is even, by requiring that for any smooth family of isospectral semiclassical potentials $V^t(x, \hbar)$, one has

$$\frac{d^k}{dt^k}\bigg|_{t=0} V_j^t = 0$$

for all $k \geqslant 1$ and all $j \geqslant 0$. And one can show by using Theorem 3.4 that $V(x, \hbar)$ is formally spectral rigid if

(1) V_0 is formally spectrally rigid, e.g. it satisfies the hypotheses of Theorem 4.4 above.
(2) For each $j \geqslant 1$, V_j satisfies the following condition: the only even function W such that

$$\int f\big(|z|^2\big)\big(V^{\mathrm{ave}}\big)^l W^{\mathrm{ave}}\, dz\, d\bar{z} = 0$$

holds for all $f \in C_0^\infty(\mathbb{R})$ and $l \in \mathbb{N}$ is the zero function.

4.2. *Recovering one dimensional even potentials*

According to (3.6), if V is an analytic function of one variable, then after averaging the only terms that survive are the even terms in the Taylor expansion of V. It follows that if $n = 1$ the semiclassical spectrum of S determines the even part of V for analytic functions V. The following theorem shows that the analyticity assumption can be dropped:

Theorem 4.5. *For $n = 1$, the first band invariant determines the even part of V, i.e. $V(x) + V(-x)$, for any smooth perturbation V.*

Proof. It is sufficient to prove the statement for V even. Then from the first band invariant one can get, for any $r > 0$, the integral

$$\int_0^{\pi/2} V(r\cos\theta)\,d\theta.$$

Making the change of variables $s = r\cos\theta$ and $u = s^2$, and denoting $V_1(x) = V(\sqrt{x})/\sqrt{x}$, the above integral becomes

$$\int_0^r V(s)\frac{1}{\sqrt{r^2 - s^2}}\,ds = \int_0^{r^2} V(\sqrt{u})\frac{1}{\sqrt{r^2 - u}}\frac{du}{\sqrt{u}} = \int_0^{r^2} V_1(u)\big(r^2 - u\big)^{-1/2}\,du.$$

The latter one equals

$$\Gamma\left(\frac{1}{2}\right)\big(J^{\frac{1}{2}}V_1\big)\big(r^2\big),$$

where $J^{\frac{1}{2}}$ is the fractional derivative of order $\frac{1}{2}$. So if we apply $J^{\frac{1}{2}}$ again and integrate, we can recover V_1, and thus V itself. □

Remark 4.6. In [8], we showed that by using semiclassical invariants modulo $O(\hbar^4)$, one can spectrally determine not only $V(x) + V(-x)$, but also $V^2(x) + V^2(-x)$. It follows that one can determine $V(x)$ itself under suitable symmetry conditions. It turns out the same result holds in higher dimensions, i.e. one can spectrally determine the integral $\int V(x)\,d\sigma_x$ as well as the integral $\int V^2(x)\,d\sigma_x$ over any sphere $|x| = r$. In particular, one can distinguish radially symmetric potentials from other potentials. For more details, cf. [7].

4.3. Recovering one dimensional odd analytic potentials

Now assume V is a one dimensional odd analytic potential,

$$V(x) = a_1x + a_3x^3 + \cdots.$$

Then

$$V^\Delta(z) = -\frac{1}{4\pi}\int_0^{2\pi}\int_0^u \left\{\sum_{k\text{ odd}} a_k\left(\frac{e^{is}z + e^{-is}\bar{z}}{2}\right)^k, \sum_{l\text{ odd}} a_l\left(\frac{e^{iu}z + e^{-iu}\bar{z}}{2}\right)^l\right\}ds\,du.$$

From the first band invariant for odd potentials (see Theorem 3.4),

$$\int e^{-\mu|z|^2} V^{\Delta}\, dz\, d\bar{z},$$

we get, by replacing μ by $\frac{\mu}{\lambda^2}$ and making change of variables $z \to \lambda z$, the spectrally determined expression

$$\sum_{k,l \text{ odd}} a_k a_l \lambda^{k+l+2} \int_{\mathbb{C}} e^{-\mu|z|^2} \int_0^{2\pi} \int_0^{u} \left\{ \left(e^{is} z + e^{-is}\bar{z}\right)^k, \left(e^{iu} z + e^{-iu}\bar{z}\right)^l \right\} ds\, du\, dz\, d\bar{z}. \tag{4.3}$$

Note that in complex coordinates,

$$\{f, g\} = \Im \frac{\partial f}{\partial z} \frac{\partial g}{\partial \bar{z}},$$

so we have for $k = 1$ and $l = 2l_1 + 1$ odd,

$$\begin{aligned}
&\int_{\mathbb{C}} e^{-\mu|z|^2} \int_0^{2\pi} \int_0^{u} \left\{ e^{is} z + e^{-is}\bar{z}, \left(e^{iu} z + e^{-iu}\bar{z}\right)^l \right\} ds\, du\, dz\, d\bar{z} \\
&\quad = \int_{\mathbb{C}} e^{-\mu|z|^2} \int_0^{2\pi} \int_0^{u} \Im l e^{i(s-u)} \left(e^{iu} z + e^{-iu}\bar{z}\right)^{l-1} ds\, du\, dz\, d\bar{z} \\
&\quad = l \binom{l-1}{(l-1)/2} \int_{\mathbb{C}} e^{-\mu|z|^2} |z|^{l-1}\, dz\, d\bar{z} \int_0^{2\pi} \int_0^{u} \sin(s-u)\, ds\, du \\
&\quad = -2\pi^2 l \binom{l-1}{l_1} \frac{l_1!}{\mu^{l_1+1}} \\
&\quad = -\frac{2\pi^2 l!}{l_1! \mu^{l_1+1}}.
\end{aligned}$$

Similarly for $k = 2k_1 + 1$ and $l = 1$ we have

$$\int_{\mathbb{C}} e^{-\mu|z|^2} \int_0^{2\pi} \int_0^{u} \left\{ \left(e^{is} z + e^{-is}\bar{z}\right)^k, e^{iu} z + e^{-iu}\bar{z} \right\} ds\, du\, dz\, d\bar{z} = -\frac{2\pi^2 k!}{k_1! \mu^{k_1+1}}.$$

In particular, we get from the lowest order term of λ in (4.3) (i.e. the coefficient of λ^4) the number $\frac{2\pi^2}{\mu} a_1^2$ from which one can recover a_1 up to a sign. At this point we must distinguish two cases.

Case 1: $a_1 \neq 0$. Suppose we have recovered the first m coefficients $a_1, \ldots, a_{2m-1}$, up to an overall sign (the same for all). Then by looking at the coefficient of λ^{2m+4} in the expression (4.3) one can recover the number $a_1 a_{2m+1} + a_{2m+1} a_1$, since all other products of pairs are known precisely. It follows that a_{2m+1} is determined up to the same sign ambiguity as a_1.

Case 2: $a_1 = 0$. More generally, suppose we have already found that $a_1 = \cdots = a_{k-2} = 0$, where $k \geqslant 3$ is an odd number. Let $l \geqslant k$ be odd too. Then a calculation as above shows that

$$\int_{\mathbb{C}} e^{-\mu|z|^2} \int_0^{2\pi} \int_0^u \{(e^{is}z + e^{-is}\bar{z})^k, (e^{iu}z + e^{-iu}\bar{z})^l\} \, ds \, du \, dz \, d\bar{z}$$
$$= C \sum_{m=0}^{k-1} \frac{1}{k-2m} \binom{k-1}{k-1-m} \binom{l-1}{\frac{l-k+2m}{2}},$$

where C is some non-vanishing constant that depends on μ. Note that

$$\frac{1}{k} \binom{k-1}{k-1} \binom{l-1}{\frac{l-k}{2}} > 0$$

and for $0 < m < \frac{k}{2}$,

$$\frac{1}{k-2m} \binom{k-1}{k-1-m} \binom{l-1}{\frac{l-k+2m}{2}} + \frac{1}{2m-k} \binom{k-1}{m-1} \binom{l-1}{\frac{l+k-2m}{2}} > 0,$$

so the expression

$$\int_{\mathbb{C}} e^{-\mu|z|^2} \int_0^{2\pi} \int_0^u \{(e^{is}z + e^{-is}\bar{z})^k, (e^{iu}z + e^{-iu}\bar{z})^l\} \, ds \, du \, dz \, d\bar{z}$$

is non-vanishing as well. It follows that for $l = k$, one can determine a_k^2, and for $l > k$, one can determine $a_k a_l$, from which one can determine a_l up to the same sign ambiguity as a_k.

In conclusion, we have proved:

Theorem 4.7. *For any odd analytic potential V, in one dimension, the first band invariant for odd potentials determines $V(x)$ up to a sign.*

4.4. Recovering constant and quadratic terms

In all dimensions $V(0)$ is spectrally determined, since

$$V(0) = V^{\text{ave}}(0) = \lim_{r \to 0} \int_{|x|^2+|p|^2=r^2} V^{\text{ave}} \, d\lambda.$$

About recovering the quadratic terms, we have already seen that we can only hope to recover the eigenvalues of the Hessian at the origin. Since the average of any odd function vanishes, we can assume that $V(x)$ is even itself. We will assume moreover that $V(x)$ is analytic, so that up to a rotation,

$$V(x) = V(0) + \sum a_i x_i^2 + \widetilde{V}(x),$$

where $\widetilde{V}(x)$ contains only terms that are homogeneous in x of degree greater than four. It follows that

$$V^{\mathrm{ave}}(x) = V(0) + \sum a_i |z_i|^2 + \widetilde{V}^{\mathrm{ave}}(z),$$

where $\widetilde{V}^{\mathrm{ave}}(z)$ contains only terms that are homogeneous in z of degree 4 and higher. So from the spectral invariant

$$\int e^{-\mu|z|^2} V_{\mathrm{ave}}^{k+1}\, dz\, d\bar{z}$$

we get, by replacing μ by $\frac{\mu}{\lambda^2}$ and making change of variables $z_i \to \lambda z_i$, the expressions

$$\lambda^{2k+4} \int e^{-\mu|z|^2} \Bigl(\sum a_i |z_i|^2\Bigr)^{k+1} dz\, d\bar{z} + \text{terms involves at least } \lambda^{2k+6}.$$

It follows that the integral

$$\int e^{-\mu|z|^2} \Bigl(\sum a_i |z_i|^2\Bigr)^{k+1} dz\, d\bar{z}$$

is spectrally determined for all k. In particular, one can determine

$$\int e^{-\mu|z|^2} e^{\sum a_i |z_i|^2}\, dz\, d\bar{z} = \pi^n \frac{1}{\mu - a_1} \cdots \frac{1}{\mu - a_n},$$

and thus determine the polynomial

$$(\mu - a_1) \cdots (\mu - a_n).$$

This proves:

Theorem 4.8. *For analytic potentials the eigenvalues of the Hessian at the origin are spectrally determined.*

4.5. Recovering the linear term for certain potentials

First assume V is an analytic odd potential, so that

$$V(x) = V_1(x) + V_3(x) + \cdots,$$

where V_k is homogeneous of degree k, and in particular, $V_1(x) = \nabla V(0) \cdot x$. Then according to Theorem 3.4, the integrals

$$\int e^{-\mu|z|^2} V^{\Delta}\, dz\, d\bar{z}$$

are spectrally determined. Again we replace μ by $\frac{\mu}{\lambda^2}$ and make change of variables $z \to \lambda z$, we get the expression

$$\lambda^2 \int e^{-\mu|z|^2} V_1^{\Delta}\, dz\, d\bar{z} + \text{terms involves at least } \lambda^4.$$

So one can spectrally determine the integral

$$\int e^{-\mu|z|^2} V_1^{\Delta}\, dz\, d\bar{z}$$

from which one can read off $\|\nabla V(0)\|^2$.

One can improve this result slightly by considering analytic potentials of the form

$$V(x) = V_0(x) + V_1(x) + V_3(x) + V_5(x) + V_6(x) + \cdots, \tag{4.4}$$

i.e. analytic potentials with vanishing quadratic and quartic terms. Note that this property,

$$V_2 = 0 \quad \text{and} \quad V_4 = 0,$$

is a spectrally property: We have already seen in Section 4.4 that one can determine whether $V_2 = 0$ using spectral data. In this class of potentials, one can spectrally determine the integrals

$$\int_{S_r} \left(V_4^{\text{ave}}\right) d\lambda.$$

If all these integrals vanish, then we get

$$\int_{S_r} e^{itV_4^{\text{ave}}}\, d\lambda = \text{Vol}(S_r),$$

from which one can deduce that $V_4^{\text{ave}} = 0$, and thus $V_4 = 0$.

Now suppose $V(x)$ is a potential of the form (4.4). Without loss of generality, one can assume that the constant term is zero. Then according to Theorem 3.3, with $l = 0$ and $f(t) = e^{-\mu t}$, one can spectrally determine

$$\int \left[e^{-\mu|z|^2} V^{\Delta} + B_2\left(e^{-\mu|z|^2}, V^{\text{ave}}\right)\right] dz\, d\bar{z}.$$

Since $V_2^{\text{ave}} = V_4^{\text{ave}} = 0$, after replacing μ by $\frac{\mu}{\lambda^2}$ and making change of variables $z \to \lambda z$, the lowest order term is again

$$\lambda^2 \int e^{-\mu|z|^2} V_1^{\Delta}\, dz\, d\bar{z},$$

and from this one gets $\|\nabla V(0)\|^2$. In conclusion, we have

Theorem 4.9. *Let $V(x)$ be an analytic potential. Then one can spectrally determine whether $V(x)$ is of the form* (4.4)*, and if V is in this class, one can spectrally determine its linear term $V_1(x)$ up to rotation.*

5. The first band invariant in dimension 2

The averaging procedure on the space of functions

$$C^\infty_{\text{even}}(\mathbb{R}^2) := \{V(x_1, x_2) = \widetilde{V}(x_1^2, x_2^2),\ \widetilde{V} \in C^\infty(\mathbb{R}^2)\}$$

of smooth functions on $\mathbb{R}^2$ which are even in each variable has particularly nice properties that we investigate in this section.

Let π be the cotangent fibration $\pi : \mathbb{C}^2 = T^*\mathbb{R}^2 \to \mathbb{R}^2$, and let V^{ave} be the average of $\pi^* V$ with respect to the circle action

$$e^{i\theta} z = (e^{i\theta} z_1, e^{i\theta} z_2). \tag{5.1}$$

As we have seen, for $\varphi \in C^\infty(\mathbb{R})$, the integrals

$$\varphi_\mu(V) = \int e^{-\frac{\mu}{2}(|z_1|^2+|z_2|^2)} \varphi(V^{\text{ave}})\, dz\, d\bar{z} \tag{5.2}$$

are spectral invariants of the operator (1.2).

To analyze these invariants we'll begin by decomposing V^{ave} into its Fourier coefficients with respect to the circle action

$$e^{i\theta} z = (e^{i\theta} z_1, e^{-i\theta} z_2), \tag{5.3}$$

and this we'll do by first examining the one dimensional analogues of these Fourier coefficients. More explicitly, let

$$B_r : C^\infty_{\text{even}}(\mathbb{R}) \to C^\infty_{\text{even}}(\mathbb{R}) \tag{5.4}$$

be the operator

$$f(x) = f\left(\frac{z+\bar{z}}{2}\right) = g(|z|^2 e^{2i\theta}) \mapsto g_r,$$

where g_r is the $2r$-th Fourier coefficient of $g(|z|^2 e^{2i\theta})$ with respect to θ. By definition this operator commutes with the homothety $x \to \lambda x$, $\lambda \in \mathbb{R}_+$, and hence maps x^{2k} into a multiple, $\gamma_{k,r} x^{2k}$, of itself. To compute $\gamma_{k,r}$ we note that

$$\left(\frac{z+\bar{z}}{2}\right)^{2k} = \frac{1}{4^k} \sum_{r=-k}^{r=k} \binom{2k}{r+k} |z|^{2k} e^{2ir\theta}.$$

Hence

$$B_r x^{2k} = \frac{1}{4^k} \binom{2k}{r+k} x^{2k} \tag{5.5}$$

for $|r| \leqslant k$ and zero for $|r| > k$. Note that B_0 is invertible.

To describe the asymptotic dependence of $\gamma_{k,r}$ on k for $k \gg 0$ we apply Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + a_1 n^{-1} + a_2 n^{-2} + \cdots\right)$$

to the quotient

$$\begin{aligned}
\frac{1}{4^k}\binom{2k}{k+r} &= 4^{-k}\frac{(2k)!}{(k+r)!(k-r)!} \\
&= \frac{4^{-k}\sqrt{4\pi k}(2k/e)^{2k}}{\sqrt{4\pi^2(k^2-r^2)}(\frac{k+r}{e})^{k+4}(\frac{k-r}{e})^{k-r}}\left(1 + b_{1,r}k^{-1} + \cdots\right) \\
&= \frac{1}{\sqrt{\pi k}}\left(1 + c_{1,r}k^{-1} + \cdots\right),
\end{aligned}$$

giving the asymptotic expansion

$$\gamma_{k,r} \sim \frac{1}{\sqrt{\pi k}}\sum_{l=0}^{\infty} c_{l,r}k^{-l}, \quad c_{0,r} = 1, \tag{5.6}$$

or (more intrinsically) the asymptotic expansion

$$B_r = \sum_{r=0}^{\infty} c_{k,r} Q^{-r-\frac{1}{2}} \tag{5.7}$$

where $Q = \frac{x}{2}\frac{d}{dx}$. Hence in particular B_r is a pseudodifferential operator on $C^\infty_{\mathrm{even}}(\mathbb{R})$ of order $-\frac{1}{2}$.

Coming back to the problem in two dimensions that prompted these computations, for each $V \in C^\infty_{\mathrm{even}}(\mathbb{R}^2)$ let $R_n V$ the nth Fourier coefficient of V^{ave} with respect to the circle action (5.3). We will prove:

Theorem 5.1. *The operator*

$$R_n : C^\infty_{\mathrm{even}}\left(\mathbb{R}^2\right) \to C^\infty_{\mathrm{even}}\left(\mathbb{R}^2\right) \tag{5.8}$$

is zero for $n \neq 0 \bmod 4$, and modulo the identification $C^\infty_{\mathrm{even}} = C^\infty_{\mathrm{even}}(\mathbb{R}) \hat{\otimes} C^\infty_{\mathrm{even}}(\mathbb{R})$,

$$R_n = B_r \hat{\otimes} B_r, \quad n = 4r \tag{5.9}$$

(i.e. for functions of the form $f(x_1,x_2) = g(x_1)h(x_2)$, $R_n f = B_r g(x_1) B_r h(x_2)$). In particular R_0 is invertible.

Proof. It suffices to check this for $g = x_1^{2k}$ and $h = x_2^{2l}$ in which case

$$\begin{aligned}
V^{\mathrm{ave}} &= \frac{1}{2\pi} 4^{-(k+l)} \int \left(z_1 e^{i\theta} + \bar{z}_1 e^{-i\theta}\right)^{2k} \left(z_2 e^{i\theta} + \bar{z}_2 e^{-i\theta}\right)^{2l} \, d\theta \\
&= \frac{1}{2\pi} 4^{-k-l} \sum_{s,t} \binom{2k}{s} \binom{2l}{t} z_1^s \bar{z}_1^{2k-s} z_2^t \bar{z}_2^{2l-t} \int e^{2i(s+t-k-l)\theta} \, d\theta \\
&= 4^{-k-l} \sum_{s+t=k+l} \binom{2k}{s} \binom{2l}{t} z_1^s \bar{z}_1^{2k-s} z_2^t \bar{z}_2^{2l-t} \\
&= \sum_{|r| \leqslant \min(k,l)} V_{k,l,r},
\end{aligned}$$

where, in polar coordinates $z_j = |z_j| e^{i\theta_j}$,

$$V_{k,l,r} = 4^{-(k+l)} \binom{2k}{k+r} \binom{2l}{l-r} |z_1|^{2k} |z_2|^{2r} e^{2ir(\theta_1 - \theta_2)}. \tag{5.10}$$

Hence setting $x_j = |z_j|$ we get for the n-th Fourier coefficient, $n = 4r$, of V^{ave} with respect to the circle action (5.3),

$$\left(\gamma_{k,r} x_1^k\right)\left(\gamma_{l,r} x_2^l\right) = B_r x_1^k B_r x_2^l. \qquad \square$$

Remark 5.2. The operator $A_r = R_n$, $n = 4r$, can also be described as a Radon transform. Namely consider the double fibration

$$\begin{tikzcd}
& \mathbb{C} \times \mathbb{C} \arrow[dl, "\pi"'] \arrow[dr, "\rho"] & \\
\mathbb{R} \times \mathbb{R} & & \mathbb{R} \times \mathbb{R}
\end{tikzcd}$$

where $\pi(z_1, z_2) = (\operatorname{Re} z_1, \operatorname{Re} z_2)$ and $\rho(z_1, z_2) = (|z_1|, |z_2|)$. Then $A_r : C^\infty_{\mathrm{even}}(\mathbb{R}^2) \to C^\infty_{\mathrm{even}}(\mathbb{R}^2)$ is the transform

$$A_r V = \rho_* \left(\int \tau^*_{\theta_1,\theta_2} \pi^* V e^{-4r\theta_2} \, d\theta_1 \, d\theta_2 \right), \tag{5.11}$$

where $\tau_{\theta_1,\theta_2}(z_1, z_2) = e^{i\theta_1}(e^{i\theta_2} z_1, e^{-i\theta_2} z_2)$.

If we take $\varphi(t) = t^m$ in (5.1) we get the spectral invariants

$$\int e^{-\frac{\mu}{2}(|z_1|^2 + |z_2|^2)} \left(V^{\mathrm{ave}}\right)^m \, dz \, d\bar{z} \tag{5.12}$$

and if we expand V^{ave} in its Fourier series with respect to the circle action (5.3), i.e. express V^{ave} as the sum

$$\sum A_r V(x_1, x_2) e^{4ri\theta},$$

then (5.12) becomes

$$\int e^{-\frac{\mu}{2}(x_1^2+x_2^2)} \left(\sum_{r_1+\cdots+r_m=0} A_{r_1} V \cdots A_{r_m} V \right) dx_1\, dx_2. \tag{5.13}$$

In particular for $m = 1$ we obtain

$$\int e^{-\frac{\mu}{2}(x_1^2+x_2^2)} A_0 V \, dx_1\, dx_2, \tag{5.14}$$

and for $m = 2$

$$\int e^{-\frac{\mu}{2}(x_1^2+x_2^2)} \sum_r |A_r V|^2 \, dx_1\, dx_2. \tag{5.15}$$

6. Inverse spectral results in two dimensions

Throughout this section we work in two dimensions, and consider the inverse spectral problem for a perturbation of the harmonic oscillator by a potential (at times semiclassical).

6.1. Results for smooth perturbations

6.1.1. Potentials of the form $f_1(x_1^2) + f_2(x_2^2)$

In this subsection we consider potentials of the form

$$V(x_1, x_2) = f_1\big(x_1^2\big) + f_2\big(x_2^2\big). \tag{6.1}$$

It is easy to see that

$$V^{\text{ave}}(x, p) = \varphi_1\big(x_1^2 + p_1^2\big) + \varphi_2\big(x_2^2 + p_2^2\big), \tag{6.2}$$

where

$$\varphi_j(r) = \frac{\Gamma(1/2)}{\pi} J^{1/2}\left(\frac{f_j(s)}{\sqrt{s}}\right)(r)$$

(see the proof of Theorem 4.5).

Theorem 6.1. *Generically, potentials of the form* (6.1) *are spectrally determined, up to the obvious symmetries*: *Exchanging of the roles of x_1 and x_2, and adding a constant to f_1 and subtracting the same constant from f_2.* (*The genericity condition is* (6.5).)

Proof. As we have seen (cf. (3.5)), the integrals

$$\int_{S_r} \left(V^{\mathrm{ave}}\right)^k(\theta)\, d\theta, \tag{6.3}$$

where $S_r \subset \mathbb{R}^4$ is the sphere of radius r, are spectrally determined. We claim that the previous integral equals

$$2\pi^2 r^3 \int_0^1 \left[\varphi_1\left(r^2(1-u)\right) + \varphi_2\left(r^2 u\right)\right]^k du. \tag{6.4}$$

To see this, consider the map $\Phi : \mathbb{R}^4 \to \mathbb{R}^2_+$ given by

$$\Phi(x, p) = \left(x_1^2 + p_1^2,\, x_2^2 + p_2^2\right) =: \left(r^2 u_1, r^2 u_2\right).$$

This maps the sphere S_r onto the line segment $u_1 + u_2 = 1$ (in the first quadrant). We will take $u = u_2$ as a coordinate in that segment. The fibers of Φ are tori, which we can parametrize in complex coordinates by $(r\sqrt{u_1}e^{is}, r\sqrt{u_2}e^{it})$. The variables (s, t, u) parametrize S_r. Using these variables to compute (6.3) yields (6.4) (we'll omit the details of the calculation).

The integrals (6.4) determine the distribution function of

$$\psi_r(u) = \varphi_1\left(r^2(1-u)\right) + \varphi_2\left(r^2 u\right)$$

on [0, 1]. We now make the genericity assumption that

$$\psi_r(u) \text{ is monotone for each } r > 0 \tag{6.5}$$

and therefore its distribution function determines it up to the ambiguity $\psi_r(1-u)$, which amounts to switching the roles of x_1 and x_2. Finally, it is not hard to see that knowing the two-variable function ψ determines φ_1 and φ_2 up to the ambiguity of adding and subtracting a constant. □

6.1.2. Semiclassical Potentials with quadratic V_0

Theorem 6.2. *Semiclassical potentials of the form*

$$V(x, \hbar) = a x_1^2 + b x_2^2 + \hbar V_1 + \hbar^2 V_2 + \cdots,$$

where $a \neq b$ and $V_i \in C^\infty_{\mathrm{even}}(\mathbb{R}^2)$ for each i, are spectrally determined.

Proof. According to Theorem 3.5, one can spectrally determine

$$\int e^{-\lambda(|z_1|^2+|z_2|^2)}\left(a|z_1|^2 + b|z_2|^2\right)^l V_1^{\mathrm{ave}}\, dz\, d\bar{z}$$

plus a term not depending on V_i with $i \geqslant 1$. We can rewrite this invariant as

$$\int e^{-\lambda(x_1^2+x_2^2)}\left(ax_1^2+bx_2^2\right)^l A_0 V_1\,dx_1\,dx_2.$$

From this we can determine

$$\int e^{-\lambda(x_1^2+x_2^2)} e^{-\mu(ax_1^2+bx_2^2)} A_0 V_1\,dx_1\,dx_2$$

for all μ and $\lambda > 0$ sufficiently large with respect to μ (so that the integral is absolutely convergent). Note that the phase equals

$$-(\lambda + a\mu)x_1^2 - (\lambda + b\mu)x_2^2,$$

and since $a \neq b$ we can make a linear change of variables, $(\lambda, \mu) \to (\mu_1, \mu_2)$ so that the invariant becomes

$$\int e^{-\mu_1 x_1^2 - \mu_2 x_2^2} A_0 V_1\,dx_1\,dx_2.$$

Integrating in polar coordinates we get, up to a universal constant, the spectral invariant

$$\int e^{-\mu_1 r_1 - \mu_2 r_2} A_0 V_1(\sqrt{r}_1, \sqrt{r}_2)/(\sqrt{r}_1\sqrt{r}_2)\,dr_1\,dr_2.$$

This function of μ_1 and μ_2 is known spectrally in some infinite wedge of $\mathbb{R}_+^2$ (determined by the condition that $\lambda > 0$). However, it is easy to see that this is an analytic function of (μ_1, μ_2), and therefore it is spectrally determined in the whole quadrant $\mathbb{R}_+^2$. Using the inverse Laplace transform, one can determine $A_0 V_1(x_1, x_2)$ pointwise, and thus determine V_1 itself.

By an inductive argument, one can similarly spectrally determine each of the remaining V_k. $\square$

6.2. Inverse spectral results for real analytic perturbations

6.2.1. Spectral rigidity of the quadratic potentials $ax_1^2 + bx_2^2$

In this section we will prove the following result, which will be used later:

Proposition 6.3. *If $a \neq b$ the potential $V = ax_1^2 + bx_2^2$ is "formally" spectrally rigid; i.e. if V_t, $-\varepsilon < t < \varepsilon$, is a smooth family of potentials each of which is an even function of x_1 and x_2 and has the same spectrum as V, then*

$$\frac{d^k}{dt^k} V_t \bigg|_{t=0} = 0 \tag{6.6}$$

for all k.

Proof. We first note that if (6.6) is non-zero for some k then we can reparametrize t so that (6.6) is non-zero for $k=1$. Thus it suffices to prove that the function $W = \frac{dV}{dt}|_{t=0}$ is zero. To see this we first note that by Remark 5.2, $A_i V = 0$ for $i \neq 0$ and $A_0 V = V$. Thus by inserting V_t into (5.13), differentiating with respect to t and setting $t=0$, we get

$$\int e^{-\frac{\mu}{2}(x_1^2+x_2^2)} V^{m-1} A_0 W \, dx_1 \, dx_2 = 0$$

for all m. Thus

$$\int e^{-\frac{\mu}{2}(x_1^2+x_2^2)} e^{-\frac{\nu}{2}V} A_0 W \, dx_1 \, dx_2 = 0$$

and since

$$-\frac{\mu}{2}\left(x_1^2+x_2^2\right) - \frac{\nu}{2}V = -\frac{\mu+\nu a}{2}x_1^2 - \frac{\mu+\nu b}{2}x_2^2$$

and $a \neq b$, we can, by making a linear change of coordinates, $(\mu,\nu) \to (\mu_1,\mu_2)$, convert this equation into the form

$$\int e^{-\frac{\mu_1}{2}x_1^2 - \frac{\mu_2}{2}x_2^2} A_0 W\left(x_1^2, x_2^2\right) dx_1 \, dx_2 = 0$$

for all μ_1, μ_2 in an infinite wedge of the 1st quadrant. Arguing just as in the end of the proof of Theorem 6.2 we can apply the inverse Laplace transform and conclude that $A_0 W = 0$, and by the injectivity of A_0, that $W = 0$. $\square$

6.2.2. Spectral determinacy

Since one can spectrally determine the integrals of $\sum |A_i V|^2$ over the circles $x_1^2 + x_2^2 = t$, in particular one can determine whether this sum vanishes to infinite order at $t = x_1 = x_2 = 0$, and hence whether V itself vanishes to infinite order at $t=0$. This can be strengthened, as follows:

Theorem 6.4. *If V is an analytic function on $\mathbb{R}^2$ such that its Taylor expansion at the origin is of the form*

$$V = V_2 + V_4 + V_6 + \cdots$$

with each V_j homogeneous of degree j, and the quadratic term V_2 has two distinct eigenvalues $a \neq b$, then V is spectrally determined (up to a rotation).[3]

Proof. We can assume without loss of generality that $V_2 = ax_1^2 + bx_2^2$, and let

$$V^{\mathrm{ave}} = a|z_1|^2 + b|z_2|^2 + V_4^{\mathrm{ave}} + V_6^{\mathrm{ave}} + \cdots$$

[3] Results of this nature for the Schrödinger operator, $-\hbar^2 \Delta_{\mathbb{R}^n} + V$, V real analytic, can be found in [6,5,10]. However, these results require strong "non-rationality" assumptions on the coefficients of the leading term $\sum a_i x_i^2$ of V.

be the Taylor series expansion of V^{ave}. From the spectral invariants

$$\int e^{-\frac{\mu}{2}(|z_1|^2+|z_2|^2)}\big(V^{\mathrm{ave}}\big)^{k+1}(z_1,z_2)\,dz\,d\bar{z}$$

we get, by replacing μ by $\frac{\mu}{\lambda^2}$ and making change of variables $z_i \to \lambda z_i$, the expressions

$$\lambda^{2k+4}\int e^{-\frac{\mu}{2}(|z_1|^2+|z_2|^2)}\big(a|z_1|^2+b|z_2|^2\big)^{k+1}\,dz\,d\bar{z}$$
$$+ (k+1)\lambda^{2k+6}\int e^{-\frac{\mu}{2}(|z_1|^2+|z_2|^2)}\big(a|z_1|^2+b|z_2|^2\big)^{k}V_4^{\mathrm{ave}}\,dz\,d\bar{z}+O\big(\lambda^{2k+8}\big)$$

and hence, from the proof of the spectral rigidity of $ax_1^2+bx_2^2$ and linearity of this expression in V_4^{ave}, this suffices to determine V_4^{ave}.

Continuing: the next unknown term in the expansion above is

$$(k+1)\lambda^{2k+8}\int e^{-\frac{\mu}{2}(|z_1|^2+|z_2|^2)}\big(a|z_1|^2+b|z_2|^2\big)^{k}V_6^{\mathrm{ave}}\,dz\,d\bar{z}$$

and hence, by the same argument, this determines V_6^{ave} (and by induction and repetition of this argument $V_8^{\mathrm{ave}}, V_{10}^{\mathrm{ave}}, \ldots$). Thus the Taylor series of V^{ave} is spectrally determinable and hence by (3.6) so is the Taylor series of V. □

6.2.3. The case of semiclassical potentials

Now consider perturbations of S_0 by the semiclassical potentials,

$$-\frac{1}{2}\big(\hbar^2\Delta+|x|^2\big)+\hbar^2\big(V_0+\hbar V_1+\hbar^2V_2+\cdots\big),$$

where

$$V_0 = ax_1^2+bx_2^2+\text{higher order terms} \tag{6.7}$$

with $a \neq b$ as above, and all V_i's are even functions. As we have seen from the previous section that the first spectral invariant is enough to determine V_0. To determine V_i's with $i \geqslant 1$, we use the higher invariants. We have seen in Section 3.5 that the integral

$$\int e^{-\lambda(|z_1|^2+|z_2|^2)}\big(V_0^{\mathrm{ave}}\big)^k V_1^{\mathrm{ave}}\,dz\,d\bar{z}$$

is spectrally determined. If we write

$$V_j = V_0^j + V_2^j + V_4^j + \cdots$$

with each V_i^j homogeneous of degree i, then the same argument as in the previous subsection shows that

$$V_0^1, \quad V_4^0 V_0^1 + V_2^0 V_2^1, \quad \ldots, \quad \sum_{l+m=k} V_{2l}^0 V_{2m}^1$$

are spectrally determined. Since each V_k^0 is already known (as component of V_0), the above functions suffice to determine each of V_k^1, and thus determine V_1. By an inductive argument, one can determine all the V_i's.

To summarize, we have proved:

Theorem 6.5. *A semiclassical potential of the form $\sum_{k\geqslant 0} \hbar^k V_k$ where V_0 is of the form (6.7) with $a \neq b$, and, for each k V_k is analytic and satisfies $V_k(\pm x_1, \pm x_2) = V_k(x_1, x_2)$, is spectrally determined.*

Appendix A. Proof of Theorem 3.3

To prove Theorem 3.3 we must compute the first two non-trivial terms in the full symbol of the operator $f(S_\chi)W^{l+1}$.

A.1. The symbol of $f(S_\chi)$

Let $\sigma_{S_\chi} \sim H_0 + \hbar^2 s_2 + \cdots$ be the asymptotic expansion of the symbol of S_χ. Note that $s_2 = V + V^{\mathrm{ave}}$. The operator

$$U(t) = e^{itS_\chi}$$

is a pseudodifferential operator of order zero. Let

$$\sigma_U \sim u_0 + \hbar u_1 + \hbar^2 u_2 + \cdots$$

be the expansion of its symbol. Writing down the equation $-i\dot{U} = S_\chi U$ symbolically it is easy to check that $u_0 = e^{itH_0}$, and one computes that

$$-i\dot{u}_1 = B_1(H_0, u_0) + H_0 u_1, \qquad u_1|_{t=0} = 0,$$

which has as solution $u_1 = 0$. u_2 is the solution to the problem

$$-i\dot{u}_2 = B_2(H_0, u_0) + H_0 u_2 + u_0 s_2, \qquad u_2|_{t=0} = 0. \tag{A.1}$$

Using the general formula

$$B_2(H, u) = -\frac{1}{4} \sum_{|\alpha|+|\beta|=2} \frac{(-1)^{|\alpha|}}{\alpha!\beta!} \big(\partial_x^\alpha \partial_p^\beta H(x,p)\big)\big(\partial_p^\alpha \partial_x^\beta u(x,p)\big),$$

one finds:

$$B_2(H_0, u_0) = \frac{1}{4}\big(t^2 H_0 - int\big) e^{itH_0}. \tag{A.2}$$

Substituting into (A.1) and integrating one obtains:

$$u_2 = i e^{itH_0}\left(\frac{t^3}{12}H_0 + \frac{nt^2}{8i} + t\left(V + V^{\mathrm{ave}}\right)\right). \tag{A.3}$$

We can now prove the following:

Lemma A.1. *Let $\sigma_{f(S_\chi)} \sim \phi_0 + \hbar\phi_1 + \hbar^2\phi_2 + \cdots$ be the asymptotic expansion of the full symbol of $f(S_\chi)$. Then one has: $\phi_0 = f(H_0)$, $\phi_1 = 0$ and*

$$\phi_2 = \left(V + V^{\mathrm{ave}}\right) f'(H_0) - \frac{n}{8} f''(H_0) - \frac{1}{12} H_0 f'''(H_0). \tag{A.4}$$

Proof. By the Fourier inversion formula,

$$f(S_\chi) = \frac{1}{2\pi}\int U(t)\hat{f}(t)\,dt.$$

At the symbolic level this reads: $\phi_j = \frac{1}{2\pi}\int u_j(t)\hat{f}(t)\,dt$. The stated formulae follow from this and the previous calculations of the u_j, $j = 0, 1, 2$. $\square$

A.2. The end of the proof

First we need to compute the second non-trivial term in the expansion of the symbol of W^{l+1}:

Lemma A.2. *For any non-negative integer l and smooth functions w_0, w_2 on $\mathbb{R}^{2n}$, one has:*

$$\left(w_0 + \hbar^2 w_2\right)^{*(l+1)} = w_0^{l+1} + \hbar^2\left((l+1)w_0^l w_2 + \sum_{j=0}^{l-1} w_0^j\, B_2\left(w_0, w_0^{l-j}\right)\right).$$

The proof is by induction on l.

Using Lemmas A.1 and A.2 and recalling that

$$w_0 = V^{\mathrm{ave}} \quad \text{and} \quad w_2 = V^{\Delta},$$

one can easily compute the second non-trivial term in the full symbol of $f(S)W^{l+1}$. The result is precisely the function (3.13) plus

$$(l+1) f\left(\frac{|x|^2 + |p|^2}{2}\right)\left(V^{\mathrm{ave}}\right)^l V^{\Delta}.$$

This finishes the proof of Theorem 3.3.

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